

Non-perturbative scalar gauge-invariant metric fluctuations from the Ponce de León metric in the STM theory of gravity

M. Anabitarte^{1,2,a}, M. Bellini^{1,2,b}

¹ Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes 3350, 7600, Mar del Plata, Argentina

² Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)

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Abstract. We study our non-perturbative formalism to describe scalar gauge-invariant metric fluctuations by extending the Ponce de León metric.

1 Introduction

The possibility that our world may be embedded in a $(4 + d)$ -dimensional universe with more than four large dimensions has attracted the attention of a great number of researches. One of these higher-dimensional theories, where the cylinder condition of the Kaluza–Klein theory [1, 2] is replaced by the conjecture that the ordinary matter and fields are confined to a 4D subspace usually referred to as a brane is the Randall and Sundrum model [3, 4].

Another non-compact theory is the so called space–time–matter (STM) or induced matter (IM) theory. In this theory the conjecture is that the ordinary matter and fields that we observe in 4D result from the geometry of the extra dimension [5, 6]. In this framework, inflationary models induced from a 5D vacuum state, where the expansion of the universe is driven by a single scalar (inflaton) field, has been subject of great activity in the last years [7–9]. The scalar metric fluctuations related to the inflaton field fluctuations can be studied as invariant under gauge transformations in a standard 4D cosmological model [10–15], or from a 5D vacuum theory of gravity [16, 17]. These perturbations are related with energy density perturbations. They are spin-zero projections of the graviton, which only exist in non-vacuum cosmologies. The issue of gauge invariance becomes critical when we attempt to analyze how the scalar metric fluctuations ψ produced in the very early universe influence the expansion with respect to the 4D background isotropic, homogeneous and 3D spatially flat cosmological metric. From the cosmological point of view, these metric fluctuations are produced by the inflaton field fluctuations $\varphi - \langle \varphi \rangle$, which describe the quantum fluctuations of the inflaton field with

respect to the expectation value of this field on the 3D sphere: $\langle \varphi \rangle$, in the absence of metric fluctuations. In other words, from the relativistic point of view, the quantum field fluctuations of the inflaton field (which is a scalar field) induce the quantum (scalar) metric fluctuations on the background metric. From the mathematical point of view the metric fluctuations are the geometrical deformations produced by quantum field fluctuations of the inflaton field.

We consider the 5D background line element [19]

$$dS^2 = l^2 dt^2 - \left(\frac{t}{t_0} \right)^{2p} l^{\frac{2p}{p-1}} dr^2 - \frac{t^2}{(p-1)^2} dl^2, \quad (1)$$

where $dr^2 = dx^2 + dy^2 + dz^2$, and l is the non-compact extra dimension and p is a dimensionless constant. This metric is 3D spatially isotropic, homogeneous and Riemann flat: $\tilde{R}_{ABCD} = 0$ (A and B run from 0 to 4), but it is curved in four dimensions [18]. From the physical point of view, this metric represents an apparent vacuum $\tilde{G}_{AB} = 0$, which deserves interest in space–time–matter theory. The particular case in which we take a foliation $l = l_0$ it is very important for cosmology [19, 20]

$$dS^2 = l_0^2 dt^2 - \left(\frac{t}{t_0} \right)^{2p} l_0^{\frac{2p}{p-1}} dr^2, \quad (2)$$

because it describes an effective 4D universe that expands with a scale factor $a(t) \sim t^p$, with a pressure $P = \frac{(2-3p)p}{8\pi G l_0^2 t^2}$ and an energy density $\rho = \frac{3p^2}{8\pi G l_0^2 t^2}$. In particular, in the limit case in which $p \rightarrow \infty$, the metric (2) describes an inflationary expansion of the universe [21] with a vacuum dominated equation of state $P \simeq -\rho$. Other important cases are $p = 1/2, 2/3$, which describe, respectively, radiation and matter dominated universes in the absence of vacuum. In this letter we shall study a non-perturbative

^a e-mail: anabitar@mdp.edu.ar

^b e-mail: mbellini@mdp.edu.ar

formalism for gauge-invariant metric fluctuations $\psi(x)$ in the STM theory of gravity, starting with the Ponce de León metric (1).

2 Formalism

With the aim to study strong gauge-invariant (scalar) metric fluctuations, we propose the following metric:

$$dS^2 = l^2 e^{2\psi} dt^2 - \left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}} e^{-2\psi} dr^2 - \frac{t^2}{(p-1)^2} e^{-2\psi} dl^2, \quad (3)$$

where $\psi(t, x, y, z, l)$ is a quantum scalar field. This metric is a generalization for strong gauge-invariant (scalar) metric fluctuations of one previously studied in [22], which is only valid for small metric fluctuations: $e^{\pm 2\psi} \simeq 1 \pm 2\psi$. To describe the system in an apparent vacuum, we shall consider the action

$${}^{(5)}I = \int d^4x dl \sqrt{\left|\frac{{}^{(5)}g}{{}^{(5)}g_0}\right|} \left(\frac{{}^{(5)}R}{16\pi G} + \frac{1}{2} g^{AB} \varphi_{,A} \varphi_{,B} \right), \quad (4)$$

where ${}^{(5)}g$ is the determinant of the covariant metric tensor g_{AB} :

$${}^{(5)}g = \left[t \left(\frac{t}{t_0}\right)^{3p} l^{\frac{4p-1}{p-1}} \frac{e^{-3\psi}}{(p-1)} \right]^2. \quad (5)$$

2.1 Lagrange equations in a 5D apparent vacuum

The Ricci scalar, which in our case is null, being given by the expression

$$\begin{aligned} {}^{(5)}R &= 4l^{\frac{-2p}{p-1}} \left(\frac{t}{t_0}\right)^{-2p} \\ &\times e^{2\psi} \left\{ \nabla^2 \psi - (\nabla \psi)^2 + e^{-4\psi} l^{\frac{2}{p-1}} \left(\frac{t}{t_0}\right)^{2p} \right. \\ &\times \left[7(\psi_{,t})^2 - 2\psi_{,tt} - \frac{3}{t}(3p+1)\psi_{,t} \right] \\ &+ \frac{l^{\frac{2p}{p-1}}}{t^2} \left(\frac{t}{t_0}\right)^{2p} (p-1) \\ &\times \left[(p-1)\psi_{,ll} - (p-1)(\psi_{,l})^2 + \frac{3p}{l}\psi_{,l} \right] \\ &\left. - \frac{3p^2}{t^2} l^{\frac{2}{p-1}} \left(\frac{t}{t_0}\right)^{2p} [1 - e^{-4\psi}] \right\}, \quad (6) \end{aligned}$$

where $\psi_{,A} = \frac{\partial}{\partial A}$. The Lagrange equations give us the relevant equations of motion for the fields φ and ψ ,

respectively:

$$\begin{aligned} \varphi_{,tt} + \left[\frac{(3p+1)}{t} - 5\psi_{,t} \right] \varphi_{,t} \left(\frac{t}{t_0}\right)^{-2p} l^{\frac{-2}{p-1}} (\nabla^2 \varphi - \nabla \psi \cdot \nabla \varphi) \\ - \frac{l^2}{t^2} e^{4\psi} (p-1)^2 \left[\varphi_{,ll} + \left(\frac{4p-1}{p-1} l^{-1} - \psi_{,l}\right) \varphi_{,l} \right] = 0, \quad (7) \end{aligned}$$

$$\begin{aligned} &\left(\frac{\partial^{(5)}R}{\partial \psi} - 3^{(5)}R \right) \\ &- \left[\frac{1}{\sqrt{|{}^{(5)}g|}} \frac{\partial \sqrt{|{}^{(5)}g|}}{\partial x^A} \frac{\partial^{(5)}R}{\partial \psi_{,A}} + \frac{\partial}{\partial x^A} \left(\frac{\partial^{(5)}R}{\partial \psi_{,A}} \right) \right] \\ &= 8\pi G \left[5l^{-2} e^{-2\psi} (\varphi_{,t})^2 - \left(\frac{t}{t_0}\right)^{-2p} l^{\frac{-2p}{p-1}} e^{2\psi} (\nabla \varphi)^2 \right. \\ &\left. - e^{2\psi} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2 \right]. \quad (8) \end{aligned}$$

The 5D energy momentum tensor for a scalar field φ in the absence of interactions is

$$T_{AB} = \varphi_{,A} \varphi_{,B} - \frac{1}{2} g_{AB} \varphi_{,C} \varphi^{,C}, \quad (9)$$

where the components g_{AB} are given by the perturbed metric (3). Using the fact that T^t_t is given by the expression

$$\begin{aligned} T^t_t &= \frac{1}{2} \left[l^{-2} e^{-2\psi} (\varphi_{,t})^2 + \left(\frac{t}{t_0}\right)^{-2p} l^{\frac{-2p}{p-1}} e^{2\psi} (\nabla \varphi)^2 \right. \\ &\left. + e^{2\psi} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2 \right], \quad (10) \end{aligned}$$

(8) can be written in a more compact manner as

$$\begin{aligned} &\left(\frac{\partial^{(5)}R}{\partial \psi} - 3^{(5)}R \right) \\ &- \left[\frac{1}{\sqrt{|{}^{(5)}g|}} \frac{\partial \sqrt{|{}^{(5)}g|}}{\partial x^A} \frac{\partial^{(5)}R}{\partial \psi_{,A}} + \frac{\partial}{\partial x^A} \left(\frac{\partial^{(5)}R}{\partial \psi_{,A}} \right) \right] \\ &= 8\pi G [6l^{-2} e^{-2\psi} (\varphi_{,t})^2 - 2T^t_t]. \quad (11) \end{aligned}$$

Equations (7) and (11) relate the quantum field φ with (quantum gauge-invariant) scalar metric fluctuations.

2.2 5D Einstein equations on an apparent vacuum

To obtain the Einstein equations, we shall calculate the components of the Einstein tensor. Their diagonal components (we consider $G_{rr} = G_{xx} + G_{yy} + G_{zz}$) are given by

$$\begin{aligned} G_{tt} &= \frac{-3}{t^2} \left\{ 2t^2 (\psi_{,t})^2 - (3p+1)t\psi_{,t} - e^{4\psi} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0}\right)^{-2p} \right. \\ &\times t^2 [(\nabla \psi)^2 - \nabla^2 \psi] + e^{4\psi} l^2 (p-1)^2 \\ &\times \left[\psi_{,ll} - (\psi_{,l})^2 - \frac{3p}{l(p-1)} \psi_{,l} \right] - p(p-1) [e^{4\psi} - 1] \left. \right\}, \quad (12) \end{aligned}$$

$$\begin{aligned}
 G_{rr} = & \frac{-3}{l^2 t^2} \left\{ e^{-4\psi} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} [3t^2 \psi_{,tt} - 9t^2 (\psi_{,t})^2 \right. \\
 & + 5(2p+1)t\psi_{,t}] + \frac{2}{3} l^2 t^2 [(\nabla\psi)^2 - \nabla^2\psi] \\
 & + \left(\frac{t}{t_0} \right)^{2p} l^{\frac{2(p-1)}{p-1}} (p-1)^2 \\
 & \times \left[(\psi_{,l})^2 - \psi_{,l} - \frac{(p^2-1)}{l(p-1)^2} \psi_{,l} \right] \\
 & \left. + 3 \left(\frac{t}{t_0} \right)^{2p} l^{\frac{2p-1}{p-1}} p^2 [1 - e^{-4\psi}] \right\}, \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 G_{ll} = & -\frac{1}{l^2 (p-1)^2} \left\{ e^{-4\psi} [3t^2 \psi_{,tt} - 9t^2 (\psi_{,t})^2 + 15pt\psi_{,t}] \right. \\
 & + l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} t^2 [(\nabla\psi)^2 - \nabla^2\psi] \\
 & \left. + l(-6p^2 + 9p - 3)\psi_{,l} + 3p(2p-1) [1 - e^{-4\psi}] \right\}. \quad (14)
 \end{aligned}$$

Since we require the Ricci scalar (5) to be null, ${}^{(5)}R = 0$, we obtain

$$\begin{aligned}
 [1 - e^{-4\psi}] = & \frac{t^2}{3p^2} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} \left\{ \nabla^2\psi - (\nabla\psi)^2 \right. \\
 & + e^{-4\psi} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \left[7(\psi_{,t})^2 - 2\psi_{,tt} - \frac{3}{t}(3p+1)\psi_{,t} \right] \\
 & \left. + \frac{l^{\frac{2p}{p-1}}}{t^2} \left(\frac{t}{t_0} \right)^{2p} (p-1) \left[(p-1)(\psi_{,ll} - (\psi_{,l})^2) + \frac{3p}{l}\psi_{,l} \right] \right\}, \quad (15)
 \end{aligned}$$

so that, using (15) in (12)–(14), we obtain the 5D diagonal components of the Einstein tensor:

$$\begin{aligned}
 G_{tt} = & \frac{-3}{t^2} \left\{ -t^2 \frac{(p+7)}{3p} (\psi_{,t})^2 + t^2 \frac{2(p+1)}{3p} \psi_{,tt} \right. \\
 & + \frac{(3p+1)}{p} t\psi_{,t} + e^{4\psi} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} \\
 & \times t^2 [(\nabla\psi)^2 - \nabla^2\psi] + e^{4\psi} l^2 (p-1)^2 \frac{2p-1}{3p} \\
 & \left. \times \left[\psi_{,ll} - (\psi_{,l})^2 - \frac{3p}{l(2p-1)} \frac{(4p+1)}{(p-1)} \psi_{,l} \right] \right\}, \quad (16)
 \end{aligned}$$

$$\begin{aligned}
 G_{rr} = & \frac{-3}{l^2 t^2} \left\{ e^{-4\psi} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} [t^2 \psi_{,tt} - 2t^2 (\psi_{,t})^2 \right. \\
 & + (p+2)t\psi_{,t}] - \frac{l^2 t^2}{3} [(\nabla\psi)^2 - \nabla^2\psi] \\
 & \left. + \left(\frac{t}{t_0} \right)^{2p} l^{\frac{3p-1}{p-1}} (2p^2 - 3p + 1)\psi_{,l} \right\} \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 G_{ll} = & -\frac{1}{l^2 (p-1)^2} \left\{ e^{-4\psi} \left[\frac{(2-p)}{p} t^2 \psi_{,tt} + \frac{(5p-7)}{p} t^2 (\psi_{,t})^2 \right. \right. \\
 & \left. - 3 \frac{(p^2-p-1)}{p} t\psi_{,t} \right] \\
 & - \frac{(p-1)}{p} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} t^2 [(\nabla\psi)^2 - \nabla^2\psi] \\
 & \left. + l^2 \frac{(2p-1)}{p} (p-1)^2 [\psi_{,ll} - (\psi_{,l})^2] \right\}. \quad (18)
 \end{aligned}$$

The diagonal components T_{AB} of the (covariant) energy momentum tensor are (see (9))

$$\begin{aligned}
 T_{tt} = & \frac{1}{2} \left\{ (\varphi_{,t})^2 + e^{4\psi} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} \right. \\
 & \left. \times (\nabla\varphi)^2 + \frac{l^2}{t^2} e^{4\psi} (p-1)^2 (\varphi_{,l})^2 \right\}, \quad (19)
 \end{aligned}$$

$$\begin{aligned}
 T_{ll} = & \frac{1}{2} \left\{ (\varphi_{,l})^2 - \frac{t^2}{l^2} \frac{e^{-4\psi}}{(p-1)^2} (\varphi_{,t})^2 \right. \\
 & \left. + \frac{t^2}{(p-1)^2} l^{\frac{-2p}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} (\nabla\varphi)^2 \right\}, \quad (20)
 \end{aligned}$$

$$\begin{aligned}
 T_{xx} = & (\varphi_{,x})^2 + \frac{1}{2} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - \frac{1}{2} (\nabla\varphi)^2 \\
 & - \frac{1}{2} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2, \quad (21)
 \end{aligned}$$

$$\begin{aligned}
 T_{yy} = & (\varphi_{,y})^2 + \frac{1}{2} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - \frac{1}{2} (\nabla\varphi)^2 \\
 & - \frac{1}{2} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2, \quad (22)
 \end{aligned}$$

$$\begin{aligned}
 T_{zz} = & (\varphi_{,z})^2 + \frac{1}{2} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - \frac{1}{2} (\nabla\varphi)^2 \\
 & - \frac{1}{2} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2. \quad (23)
 \end{aligned}$$

Since the metric (3) is 3D spatially isotropic, we can make the identification $T_{rr} = T_{xx} + T_{yy} + T_{zz}$, and we obtain

$$\begin{aligned}
 T_{rr} = & \frac{3}{2} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - \frac{1}{2} (\nabla\varphi)^2 \\
 & - \frac{3}{2} l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2. \quad (24)
 \end{aligned}$$

Finally, using (12)–(14) with (19), (24) and (20), we obtain

$$\begin{aligned}
 & \left\{ \frac{2(p+1)}{3p} \psi_{,tt} - \frac{(p+7)}{3p} (\psi_{,t})^2 + \frac{(3p+1)}{p} \frac{1}{t} \psi_{,t} \right. \\
 & \left. + e^{4\psi} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} \frac{1-2p}{3p} [(\nabla\psi)^2 - \nabla^2\psi] \right\}
 \end{aligned}$$

$$\begin{aligned}
& + e^{4\psi} \frac{l^2}{t^2} (p-1)^2 \frac{2p-1}{3p} \left[\psi_{,ll} - (\psi_{,l})^2 - \frac{3p(4p+1)}{l(2p-1)(p-1)} \psi_{,l} \right] \Bigg\} \\
& = \frac{4\pi G}{3} \left\{ (\varphi_{,t})^2 + e^{4\psi} l^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} (\nabla\varphi)^2 \right. \\
& \quad \left. + \frac{l^2}{t^2} e^{4\psi} (p-1)^2 (\varphi_{,l})^2 \right\}, \tag{25}
\end{aligned}$$

$$\begin{aligned}
& e^{-4\psi} l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \left[\psi_{,tt} - 2(\psi_{,t})^2 + \frac{(p+2)}{t} \psi_{,t} \right] \\
& - \frac{1}{3} [(\nabla\psi)^2 - \nabla^2\psi] + \left(\frac{t}{t_0} \right)^{2p} \frac{l^{\frac{p+1}{p-1}}}{t^2} (2p^2 - 3p + 1) \psi_{,l} \\
& = \frac{4\pi G}{3} \left\{ 3l^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - (\nabla\varphi)^2 \right. \\
& \quad \left. - 3l^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2 \right\}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
& \frac{t^2}{l^2} e^{-4\psi} \left[\frac{(2-p)}{p} \psi_{,tt} + \frac{(5p-7)}{p} (\psi_{,t})^2 - \frac{3}{t} \frac{p^2-p-1}{p} \psi_{,t} \right] \\
& + \frac{(p-1)}{p} l^{\frac{-2(2p-1)}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} t^2 [(\nabla\psi)^2 - \nabla^2\psi] \\
& - \frac{(2p-1)(p-1)^2}{p} [\psi_{,ll} - (\psi_{,l})^2] \\
& = -\frac{4\pi G}{3} \left\{ (p-1)^2 (\varphi_{,l})^2 - \frac{t^2}{l^2} e^{-4\psi} (\varphi_{,t})^2 \right. \\
& \quad \left. + t^2 l^{\frac{-2p}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} (\nabla\varphi)^2 \right\}. \tag{27}
\end{aligned}$$

Equations (25), (26) and (34) give us the diagonal Einstein equations on a 5D apparent vacuum. In the following section we shall use these equations to describe the effective 4D physics on a curved 4D hypersurface that is embedded on the perturbed metric (3).

3 Effective 4D dynamics of φ and gauge-invariant metric fluctuations ψ

In order to study the effective 4D dynamics of the fields φ and ψ , we can make a foliation $l = l_0$, so that $dl = 0$, and the perturbed 4D hypersurface of (2) as a result is

$$dS^2 = l_0^2 e^{2\psi} dt^2 - \left(\frac{t}{t_0} \right)^{2p} l_0^{\frac{2p}{p-1}} e^{-2\psi} dr^2. \tag{28}$$

An interesting limit case of the metric (28) is the one in which $p \rightarrow \infty$. In such a case the metric (28) describes an asymptotical vacuum expansion (i.e., a de Sitter expansion), which should be relevant for describing the early (inflationary) universe. The particular case in which the gauge-invariant metric fluctuations are weak was studied in [22]. The effective 4D action is ${}^{(4)}I = \int d^4x {}^{(4)}L$, where

${}^{(4)}L$ is the effective 4D Lagrangian

$${}^{(4)}L = \sqrt{\left| \frac{{}^{(4)}g}{{}^{(4)}g_0} \right|} \left(\frac{{}^{(4)}R}{16\pi G} + \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - V \right) \Bigg|_{l=l_0}, \tag{29}$$

such that V is the effective 4D potential induced by the foliation $l = l_0$:

$$V = -\frac{1}{2} g^{ll} \left(\frac{\partial\varphi}{\partial l} \right)^2 \Bigg|_{l=l_0}, \tag{30}$$

and ${}^{(4)}R$ is the effective 4D Ricci scalar whose origin is also geometrically induced by the foliation on the flat metric (3)

$$\begin{aligned}
{}^{(4)}R & = \frac{2e^{-2\psi}}{l_0^2 t^2} \left\{ 3p(2p-1) - t^2 \left(\frac{t}{t_0} \right)^{-2p} l_0^{\frac{-2p}{p-1}} \right. \\
& \quad \left. \times e^{4\psi} [(\nabla\psi)^2 - \nabla^2\psi] - 3t [5p\psi_{,t} - 3t(\psi_{,t})^2 + t\psi_{,tt}] \right\}. \tag{31}
\end{aligned}$$

The effective 4D (diagonal) Einstein equations for the components tt , rr and ll are given respectively by

$$\begin{aligned}
& \left\{ -\frac{(p+7)}{3p} (\psi_{,t})^2 + \frac{2(p+1)}{3p} \psi_{,tt} + \frac{(3p+1)}{p} \frac{1}{t} \psi_{,t} \right. \\
& + e^{4\psi} l_0^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} \frac{1-2p}{3p} [(\nabla\psi)^2 - \nabla^2\psi] \\
& + e^{4\psi} \frac{l_0^2}{t^2} (p-1)^2 \frac{2p-1}{3p} \\
& \left. \times \left[\psi_{,ll} - (\psi_{,l})^2 - \frac{3p(4p+1)}{l_0(2p-1)(p-1)} \psi_{,l} \right] \right\} \Bigg|_{l=l_0} \\
& = \frac{4\pi G}{3} \left\{ (\varphi_{,t})^2 + e^{4\psi} l_0^{\frac{-2}{p-1}} \left(\frac{t}{t_0} \right)^{-2p} (\nabla\varphi)^2 \right. \\
& \quad \left. + \frac{l_0^2}{t^2} e^{4\psi} (p-1)^2 (\varphi_{,l})^2 \right\} \Bigg|_{l=l_0}, \tag{32}
\end{aligned}$$

$$\begin{aligned}
& e^{-4\psi} l_0^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \left[\psi_{,tt} - 2(\psi_{,t})^2 + \frac{(p+2)}{t} \psi_{,t} \right] \\
& - \frac{1}{3} [(\nabla\psi)^2 - \nabla^2\psi] + \left(\frac{t}{t_0} \right)^{2p} \frac{l_0^{\frac{p+1}{p-1}}}{t^2} (2p^2 - 3p + 1) \psi_{,l} \Bigg|_{l=l_0} \\
& = \frac{4\pi G}{3} \left\{ 3l_0^{\frac{2}{p-1}} \left(\frac{t}{t_0} \right)^{2p} e^{-4\psi} (\varphi_{,t})^2 - (\nabla\varphi)^2 \right. \\
& \quad \left. - 3l_0^{\frac{2p}{p-1}} \left(\frac{t}{t_0} \right)^{2p} \frac{(p-1)^2}{t^2} (\varphi_{,l})^2 \right\} \Bigg|_{l=l_0}, \tag{33}
\end{aligned}$$

$$\frac{t^2}{l_0^2} e^{-4\psi} \left[\frac{(2-p)}{p} \psi_{,tt} + \frac{(5p-7)}{p} (\psi_{,t})^2 - \frac{3}{t} \frac{p^2-p-1}{p} \psi_{,t} \right]$$

$$\begin{aligned}
& + \frac{(p-1)}{p} l_0^{\frac{-2(2p-1)}{p-1}} \left(\frac{t}{t_0}\right)^{-2p} t^2 [(\nabla\psi)^2 - \nabla^2\psi] \\
& - \frac{(2p-1)(p-1)^2}{p} [\psi_{,ll} - (\psi_{,l})^2] \Big|_{l=l_0} \\
& = -\frac{4\pi G}{3} \left\{ (p-1)^2 (\varphi_{,l})^2 - \frac{t^2}{l_0^2} e^{-4\psi} (\varphi_{,t})^2 \right. \\
& \quad \left. + t^2 l_0^{\frac{-2p}{p-1}} \left(\frac{t}{t_0}\right)^{-2p} (\nabla\varphi)^2 \right\} \Big|_{l=l_0}. \quad (34)
\end{aligned}$$

The effective Lagrangian equations are

$$\begin{aligned}
& \varphi_{,tt} + \left[\frac{(3p+1)}{t} - 5\psi_{,t} \right] \varphi_{,t} \\
& - e^{4\psi} \left(\frac{t}{t_0}\right)^{-2p} l_0^{\frac{-2}{p-1}} (\nabla^2\varphi - \nabla\psi \cdot \nabla\varphi) \\
& - \frac{l_0^2}{t^2} e^{4\psi} (p-1)^2 \left[\varphi_{,ll} + \left(\frac{(4p-1)}{(p-1)} l_0^{-1} - \psi_{,l} \right) \varphi_{,l} \right] \Big|_{l=l_0} = 0, \quad (35)
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\partial^{(4)}R}{\partial\psi} - 3^{(4)}R \right) \\
& - \left[\frac{1}{\sqrt{|^{(4)}g|}} \frac{\partial\sqrt{|^{(4)}g|}}{\partial x^A} \frac{\partial^{(4)}R}{\partial\psi_{,A}} + \frac{\partial}{\partial x^A} \left(\frac{\partial^{(4)}R}{\partial\psi_{,A}} \right) \right] \Big|_{l=l_0} \\
& = 8\pi G [6l^{-2} e^{-2\psi} (\varphi_{,t})^2 - 2T_t^t] \Big|_{l=l_0}, \quad (36)
\end{aligned}$$

which, like the Einstein equations (25), (26) and (34), are non-linear. They give us the dynamics of the system described by the fields φ and ψ .

Since the field $\varphi(t, \mathbf{r})$ is of quantum origin, it should be described by the following non-commutative algebra:

$$\begin{aligned}
& [\varphi(t, \mathbf{r}), \Pi_\varphi(t, \mathbf{r}')] \\
& = i g^{tt} \sqrt{\left| \frac{^{(4)}g}{^{(4)}g_0} \right|} e^{-\int \left[\frac{3p+1}{t} - 5\psi_{,t} \right] dt} \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (37)
\end{aligned}$$

where $|^{(4)}g| = \left(e^{-2\psi} \left(\frac{t}{t_0}\right)^{3p} l_0^{\frac{4p-1}{p-1}} \right)^2$ is the determinant of the effective 4D perturbed metric tensor $g_{\mu\nu}$. The canonical momentum $\Pi_\varphi = \frac{\partial^{(4)}L}{\partial\dot{\varphi}}$ is given by

$$\Pi_\varphi = g^{tt} \sqrt{\left| \frac{^{(4)}g}{^{(4)}g_0} \right|} \dot{\varphi}. \quad (38)$$

Of course, due to the non-linear nature of the Einstein equations, it is almost impossible to resolve the field equations without making some approximation.

4 Final comments

We have extended to the Ponce de León metric in the non-perturbative formalism proposed in [22]. To do this, we

have introduced the metric (3), which, once we take a foliation $l = l_0$, takes into account the gauge-invariant metric fluctuations during the expansion of the universe at its origin (see (28)), as a back reaction effect of the inflaton field fluctuations $\varphi - \langle\varphi\rangle$. The background 4D version of the Ponce de León metric is very important for cosmology, because it describes a power-law expansion for the universe. The interesting thing of this formalism is that the system is considered from the point of view of a 5D perturbed flat metric, on which we assume an apparent vacuum state. Hence, all 4D sources come from the geometrical foliation $l = l_0$ on the 5D metric (3) (which is Riemann flat). The advantage of this formalism with respect to another one previously introduced [21] should be in the description of the strong metric fluctuations, which should be more important in the early universe on very small scales. Of course, the results obtained in [21] should here be recovered in the weak field approximation. In this approximation back reaction effects become negligible and the 4D version of the Ponce de León metric (see (2)), describes in the limit case $p \rightarrow \infty$ a vacuum dominated expansion of the universe with an equation of state $P/\rho = \omega \simeq -1$ as reported recently in [23].

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